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One-loop effects in a self-dual planar noncommutative theory

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Abstract: We study the UV properties, and derive the explicit form of the one-loop effective action, for a noncommutative complex scalar field theory in $2 + 1$ dimensions with a Grosse-Wulkenhaar term, at the self-dual point. We also consider quantum effects around non-trivial minima of the classical action which appear when the potential allows for the spontaneous breaking of the U(1) symmetry. For those solutions, we show that the one-loop correction to the vacuum energy is a function of a special combination of the amplitude of the classical solution and the coupling constant.

Keywords: Solitons Monopoles and Instantons, Field Theories in Lower Dimensions, Non-Commutative Geometry.

Contents

1. Introduction

Noncommutative Quantum Field Theories (NCQFT's), in particular those obtained by Moyal deformation of the usual (pointwise) product of functions, have been a subject of intense research in recent years [1], because of many different reasons. Among them is their relevance to open string dynamics [2] and, in a quite different context, they are important tools for an effective description of the Quantum Hall Effect (QHE) [3]. In this realization of an incompressible quantum fluid [4], the projection to the lowest Landau level under the existence of a strong magnetic field amounts, for a two-dimensional system, to the noncommutativity of the spatial coordinates [5].

In this paper, we calculate one-loop quantum effects around both trivial and non-trivial saddle points, for the NCQFT of a self-interacting complex scalar field equipped with a Grosse-Wulkenhaar (GW) term [6] (see also [7] and [8]).

One of the interests for carrying out this explicit calculation is that, in spite of the many important general results for this kind of NCQFT [6] there are, we believe, still few concrete results obtained by actually evaluating quantum effects in models that include a GW-term. In particular, we shall focus on the divergent terms in the effective action, and on the first quantum corrections to the effective action around non trivial minima, in the case of a spontaneous symmetry breaking potential.

We deal with a $2+1$ dimensional model, something which makes it more attractive from the point of view of its potential applications to the situation of a planar system in an external magnetic field. At the same time, it provides an opportunity to probe the effect to the GW term in an odd number of spacetime dimensions where, necessarily, some of the coordinates do commute. Finally, we also consider the important issue of calculating quantum corrections on top of non-trivial minima that arise when there is spontaneous symmetry breaking.

This article is organized as follows: in section 2 we write down the action that defines the model, selecting the basis of functions to be used in the loopwise expansion, and extracting the resulting Feynman rules. We analyze the renormalizability of the theory in 3, while the one-loop corrections to the two and four-point functions are evaluated in section 4. We consider quantum effects around non-trivial minima in section 5. In section 6, we present our conclusions.

2. The model

We are concerned with a noncommutative model whose dynamical variable is a complex scalar field in $2+1$ space-time dimensions, such that the coordinates satisfy:

$$[x_\mu\,,\,x_\nu] \;=\; i\,\theta_{\mu\nu} \quad,\quad \mu,\,\nu \;=\; 0,\,1,\,2\;, \tag{2.1}$$

where $\theta_{\mu\nu}$ are the elements of a constant real antisymmetric constant matrix. In $2+1$ dimensions this matrix is necessarily singular; thus we shall assume that its (only) null eigenvalue corresponds to the time direction, x_0, since we are not interested in introducing noncommutativity for the time coordinate. Although there are some general arguments to discard those theories [9], in our case the reason is simpler: we want to consider theories that might be interpreted in terms of effective field theory models in strong magnetic fields [10]. Thus we have the more explicit commutation relations:

$$[x_0, x_j] \;=\; 0 \quad,\quad [x_j, x_k] \;=\; i\,\theta_{jk} \quad,\quad j,\,k = 1,\,2 \tag{2.2}$$

where $\theta_{jk} = \theta\,\epsilon_{jk}$, and we shall assume that $\theta > 0$.

The model is defined by the following Euclidean action:

$$\mathcal{S} = \int_{x,t} \big(\partial_\mu\varphi^*\partial_\mu\varphi + m^2\varphi^*\varphi + \Omega^2\varphi^*\star z_j\star\varphi\star z_j\big) \,+\, \mathcal{S}_{\rm int} \tag{2.3}$$

(with $z_j \equiv \theta^{-1}_{jk}x_k$), which is of the kind proposed in [6]. Under the extra assumption that $\Omega^2 \equiv 2$, the system is said to be at the self-dual point since it is invariant under a combined Fourier transformation and rescaling [11] of spatial coordinates:

$$\mathcal{S}[\varphi,\varphi^*,\theta,g] = \mathcal{S}\left[\frac{1}{\theta}\hat{\varphi}_{(\frac{x}{\theta})},\frac{1}{\theta}\hat{\varphi^*}_{(\frac{x}{\theta})},\theta,g\right]\;, \tag{2.4}$$

a symmetry that survives, as we shall see explicitly, one loop quantum corrections. At that special point, one of the terms in the commutators used to define the spatial (inner) derivatives is canceled with a like one coming from the confining potential term, leading to an action with the form:

$$\mathcal{S} = \int_{x,t} \left(\dot{\varphi}^* \dot{\varphi} + m^2 \varphi^* \varphi + \frac{1}{\theta^2} \varphi^* \star x_j \star x_j \star \varphi + \frac{1}{\theta^2} \varphi \star x_j \star x_j \star \varphi^* \right) + \mathcal{S}_{\rm int} \ , \tag{2.5}$$

where the dot denotes differentiation with respect to x_0. The interaction term that we shall consider may be regarded as the orientable analog of the φ^4 vertex, namely,

$$\mathcal{S}_{\rm int} = \frac{g}{4!} \int_{x,t} \varphi^* \star \varphi \star \varphi^* \star \varphi \ . \tag{2.6}$$

Note that there is, indeed, yet another inequivalent analog to the φ^4 vertex, namely:

$$\mathcal{S}_{\rm int} = \frac{g}{4!} \int_{x,t} \varphi^* \star \varphi^* \star \varphi \star \varphi \ . \tag{2.7}$$

We shall not, however, deal here with a theory including this term since its UV properties seem to be qualitatively different [12]. The interaction term (2.6) yields a super-renormalizable theory, as we shall see in section 3.

To carry on explicit calculations it is convenient to chose the so called *matrix basis*, since, as it can be shown, their $\star$-product adopts a 'diagonal form':

- $f_{nk} \star f_{k'n'} = \delta_{kk'} f_{nn'}$
- $(f_{nk})^* = f_{kn}$.

In appendix A, a brief summary of this and related properties is presented. Careful demostrations may be found, for example, in [13].

The coefficients $\varphi_{nk}(t)$, that appear in the expansion of the field in such a basis,

$$\varphi(x,t) \ = \sum_{n,\,k\,\geq 0} \varphi_{nk}(t) f_{nk}(x) \ , \tag{2.8}$$

become then the dynamical variables. In terms of these coefficients, the action integral reads:

$$\mathcal{S} = \int_{t_1 t_2} \varphi^*_{ln}(t_1) G_{ln,kr}(t_1 - t_2) \varphi_{kr}(t_2) + \frac{2\pi\theta g}{4!} \int_t \varphi^*_{n_1,n_4} \varphi_{n_1,n_2} \varphi^*_{n_3,n_2} \varphi_{n_3,n_4} \ , \tag{2.9}$$

where

$$G_{ln,kr}(t_1 - t_2) = 2\pi\theta \delta(t_1 - t_2) \delta_{lk} \delta_{nr} \Big(- \partial_t^2 + m^2 + \frac{2}{\theta}(k + n + 1) \Big) \tag{2.10}$$

is a kernel that defines the quadratic (free) part of the action. To derive the Feynman rules corresponding to this action, we need an explicit expression for $\Delta = G^{-1}$. Since G is already diagonal with respect to its discrete indices, we only need to deal with the temporal coordinates. In Fourier (frequency) space:

$$\hat{\Delta}_{ln,kr}(\nu) = \frac{\delta_{lk} \delta_{nr}}{2\pi\theta} \frac{1}{\omega_{nk}^2 + \nu^2} \ , \tag{2.11}$$

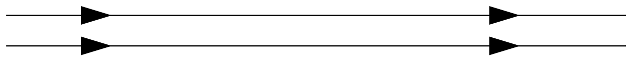

Figure 1: The free propagator

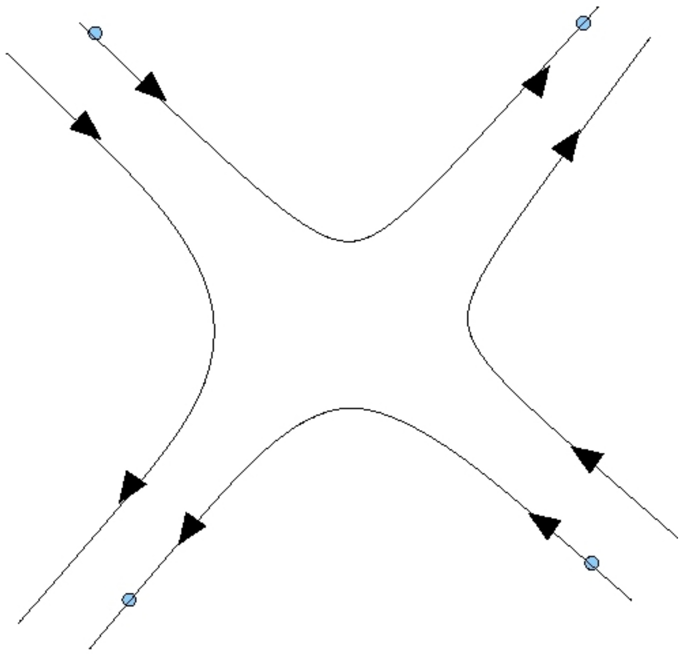

Figure 2: The interaction vertex $\frac{g}{4!}\varphi^*_{n_1,n_4}\varphi_{n_1,n_2}\varphi^*_{n_3,n_2}\varphi_{n_3,n_4}$

and after Fourier transformation:

$$\Delta_{ln,kr}(t_1-t_2) = \langle\varphi_{ln}(t_1)\varphi^*_{kr}(t_2)\rangle_0 = \frac{\delta_{lk}\delta_{nr}}{2\pi\theta}\frac{e^{-\omega_{kn}|t_1-t_2|}}{2\omega_{kn}} \quad , \tag{2.12}$$

where

$$\omega^2_{kn} = m^2 + \frac{2}{\theta}(k+n+1) \quad . \tag{2.13}$$

The Feynman rules and conventions used for the diagrammatic expansion that follows from this model are better introduced in terms of diagrams with a double line notation, to cope with matrix indices. Orientation is, on the other hand, assigned according to the usual convention for creation and annihilation operators.

The free propagator and the interaction vertex correspond to the diagrams of figures 1 and 2, respectively.

A dot attached to a line indicates that it corresponds to the first index. So when two vertices are connected with a double line, both the dots and the orientation of the lines must coincide (note that it is not necessary to attach a dot to the propagator). Equipped with this notation, we may easily group all the inequivalent diagrams corresponding to a given class. Symmetry factors can, of course, be calculated by standard application of Wick's theorem.

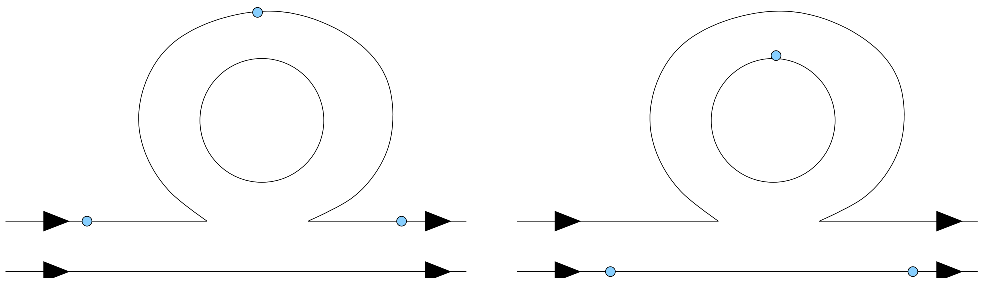

Figure 3: The tadpole graph. Two contractions are possible.

Thus we are ready to construct perturbatively the generating functional of 1PI graphs which we shall calculate explicitly up to the one-loop order.

This analysis is adapted for a propagator with a simple form in the matrix basis. Other basis can be of interest (such as plane waves) depending on the structure of the propagator and the vertex [14].

3. Renormalization

3.1 One-loop divergences

It is easy to see that the only divergent diagram of the theory at the one-loop level is the *tadpole* graph of figure 3.

As shown in the figure, there is a 'free' internal index (not fixed by the external ones). This leads to an UV divergent contribution to the two point function:

$$\Gamma^{(2),planar}_{n_0 n_1, n_2 n_3}(x-y) = \frac{2g}{4!}\delta_{n_0 n_2}\delta_{n_1 n_3}\delta(x-y)\sum_{k\geq 0}\frac{1}{\omega_{k,n_2}} \ . \tag{3.1}$$

The same amplitude is obtained writing n_3 instead of n_2 and this would yield to a symmetric expression in φ and φ^*. This corresponds to the other contraction shown in figure 3. For the sake of simplicity we concentrate now in one of these, and we finally give the symmetrized expression in equation (4.5).

An Euclidean cut off can be implemented simply by limiting the number of modes we sum. Denoting by $k_{\max}$ the maximum index in the (convergent) sum, we split it up into two parts: one of them shall give a mass renormalization term, while the other will be a function of n_2 with a finite limit as $k_{\max} \to \infty$. We chose as subtraction point $n_2 = 0$, in this way the singular contribution to Γ is:

$$\delta\Gamma = \frac{2g}{\pi\theta 4!}\left(\sum_{k=0}^{k=k_{\max}}\frac{1}{\omega_{k,0}}\right)\int_{x,t}\varphi^*_{(x,t)}\star\varphi_{(x,t)} \ , \tag{3.2}$$

which can be absorbed by the definition of the mass parameter.

On the other hand, the finite part reads:

$$\sum_{k=0}^{k=k_{\max}}\left(\frac{1}{\omega_{k,n_2}}-\frac{1}{\omega_{k,0}}\right) , \tag{3.3}$$

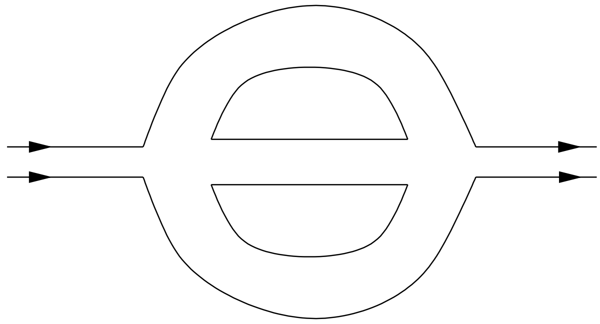

Figure 4: Two loop self energy diagram.

where the $k_{\max} \to \infty$ limit can be taken to get a finite contribution to the generating functional. This yields a function of n_2 that, as we shall see, can be written as a (one-body) potential term.

3.2 Renormalizability and power counting

Let us first show the theory is at least renormalizable (by power counting). For a given Green's function the most important contributions are given by the planar graphs. But taking into account the structure of the propagator (2.11) any amplitude must converge better than a fermionic theory with a quartic vertex two dimensions (and without infrared problems). In order to show this we recall the standard definition:

$$\omega_{\text{vertex}} = \left(\frac{d-1}{2}\right) F_\nu \quad , \tag{3.4}$$

where F_ν is the number of fermions in the vertex. So in our case the theory behaves better than $\omega_\nu = 2$, i.e. a renormalizable theory.

In order to see that the theory is super-renormalizable, note that there must be at least two propagators in each loop (in other case, it would be the one-loop tadpole contribution, that has already been considered), but products of two or more propagators of the form (2.11) yield convergent integrals, because the argument can be sum or integrated in any order and each of the iterated operation converges [15]. One way to see this is integrating in the worst iteration possible, this is to perform the continuous integral and then the sum. But if one of the propagators is multiplied by a rational function of the discrete variable the sum converges, and this is indeed the case (as can be easy verified performing the integral asymptotically).

There remain non-trivial cases, namely: overlapping loop graph such as the one shown in figure 4.

The amplitude associated with this diagram is proportional to:

$$\int_{\omega_1\omega_2} \sum_{n_1 n_2} \frac{1}{\omega_1^2 + m^2 + \frac{2}{\theta}(k_1+n_1+1)} \frac{1}{\omega_2^2 + m^2 + \frac{2}{\theta}(k_2+n_2+1)} \times \\ \frac{1}{(\omega_1+\omega_2-q)^2 + m^2 + \frac{2}{\theta}(n_1+n_2+1)} \, , \tag{3.5}$$

where k_1, k_2 and q are external variables. This graph is convergent iff the following integral is convergent:

$$\int d^4x \frac{1}{x_1^2+|x_2|+1}\frac{1}{x_3^2+|x_4|+1}\frac{1}{(x_1+x_3-\beta)^2+|x_1|+|x_2|+1} \ , \tag{3.6}$$

but this is indeed the case, because is an integral of a positive function and the integration in each variable is convergent. Any other multiloop planar diagram is convergent for the same reason. In this way we see that it is enough to renormalize the tadpole graph.

4. Renormalized generating functional

We construct here the generating functional of 1PI graphs for the one loop renormalized perturbation series up to fourth order in the field variable.

4.1 Two point function

We need to consider the expression in (3.3) in more detail. This is a convergent series which defines a holomorphic function of n_2. Introducing coefficients α_λ, so that:

$$\sum_{k\geq 0}(\frac{1}{\omega_{k,n_2}}-\frac{1}{\omega_{k,0}}) = \sum_{\lambda\geq 1}\alpha_\lambda n_2^\lambda \ , \tag{4.1}$$

the relation:

$$\alpha_\lambda = \sqrt{\frac{\theta}{2}}\beta^{(\lambda)}_{(1+\frac{m^2\theta}{2})} \tag{4.2}$$

$$\beta^{(\lambda)}(z) = \frac{1}{\lambda!}\frac{\partial^\lambda}{\partial w^\lambda}\left(\mathcal{Z}\left[\frac{1}{2},w+z\right]-\mathcal{Z}\left[\frac{1}{2},z\right]\right)|_{w=0} \tag{4.3}$$

where $\mathcal{Z}$ is the Hurwitz zeta function, is easily obtained. It is important to note the smooth behavior with respect to the product $m^2\theta$, this number is greater than zero so the argument of the function beta is always greater than one (i.e. in this domain the function is regular). Now we are ready to include the contribution of the two point function to the generating functional. The singular part is absorbed in a mass renormalization, while the finite part is:

$$\Gamma^{(2),finite}_{n_0n_1,n_2n_3}(x-y) = \frac{2g}{4!}\delta_{n_0n_2}\delta_{n_1n_3}\delta(x-y)\left(\sum_{\lambda\geq 1}\alpha_\lambda n_2^\lambda\right) \ . \tag{4.4}$$

Taking now into account the correspondence with the functional representation (see appendix A), we can use the number operator to get an expression in the original functional space:

$$\delta\Gamma[\varphi,\varphi^*] = \frac{2g}{4!2\pi\sqrt{2\theta}}\int_{x,t}\left(\varphi^*\star V\left(\frac{x}{\sqrt{\theta}}\right)\star\varphi+\varphi\star V\left(\frac{x}{\sqrt{\theta}}\right)\star\varphi^*\right) \ , \tag{4.5}$$

where we have used the definition:

$$V(\frac{x}{\sqrt{\theta}}) = \sum_{\lambda\geq 1}\frac{\beta^{(\lambda)}}{2^\lambda}\left(\frac{x_j\star x_j}{\theta}-1\right)^{\star\lambda} \ . \tag{4.6}$$

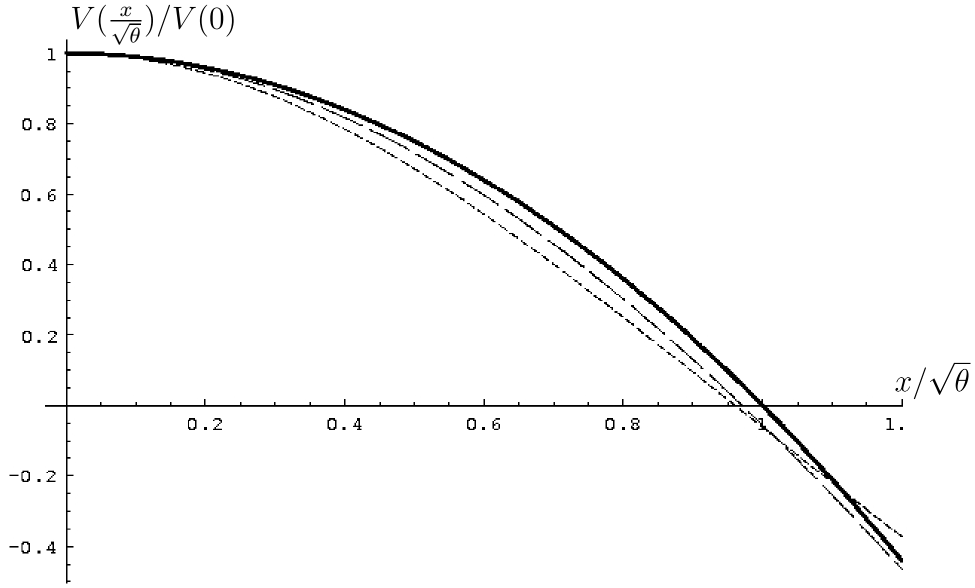

Figure 5: One-body potential due to quantum corrections. $m^2\theta = 0$ (Short dashed), $m^2\theta = 2$ (Long dashed), $m^2\theta \to \infty$ (bold).

This shows the explicit form of the one-body potential, The first three terms in the expansion of this potential are plotted in figure 5, for the values $m^2\theta = 0$, $m^2\theta = 2$ and $m^2\theta = \infty$.

It is clear that this quantum correction tends to deconfine the system, as it should be expected from the repulsive character of the interaction.

4.2 Four-point function

Now we deal with the four-point contributions, which correspond to four inequivalent diagrams, which we study below, together with their corresponding contributions to the action. The diagram of figure 6 contributes with:

$$\delta\Gamma = -\frac{S(2\pi\theta g)^2}{2(4!)^2}\delta_{(t_1-t_2)}\delta_{(t_3-t_4)}\delta^{n_1}_{n_3}\delta^{n_2}_{n_0}\delta^{n_5}_{n_7}\delta^{n_4}_{n_6}(\Delta^{n_4 n_1, n_4 n_1}_{(t_1-t_3)})^2 \quad , \tag{4.7}$$

where S is a symmetry factor. To obtain an explicit expression for the quantum correction to the action we will consider a low energy approximation, assuming we are concerned with the physics of this system up to $n_i^{\max}$ (which is a kind of low-momentum approximation).

Thus, assuming the condition $\theta m^2 >> n_i^{\max}$ for the external indices, we can write:

$$\delta\Gamma = -\alpha\frac{g^2}{m^3\theta^2}\int_t \left(\int_x \varphi^*_{(x,t)} \star \varphi_{(x,t)}\right)^2 , \quad \alpha > 0 \quad , \tag{4.8}$$

where α is independent of the parameters of the problem. Another contribution is the one represented in figure 7. Its analytic expression is:

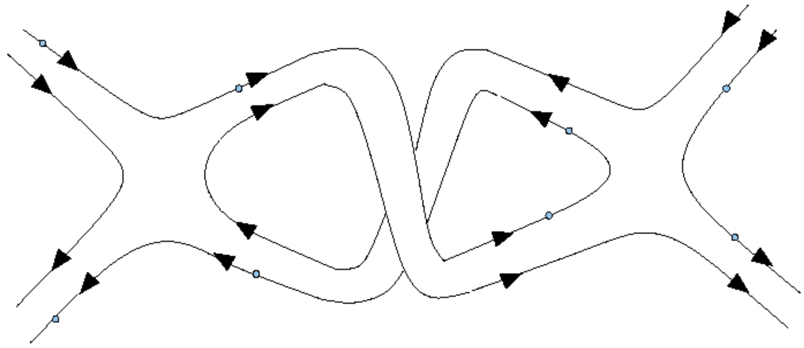

Figure 6: Non planar contribution to the four point function.

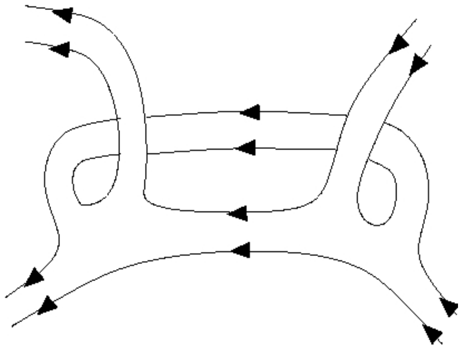

Figure 7: Non planar contribution to the four point function.

$$\delta\Gamma = \frac{-S(2\pi\theta g)^2}{2(4!)^2}\delta_{(t_1-t_3)}\delta_{(t_2-t_4)}\delta^{n_6}_{n_4}\delta^{n_3}_{n_1}\delta^{n_2}_{n_0}\delta^{n_5}_{n_7}\Delta^{n_1n_4,n_1n_4}_{(t_2-t_3)}\Delta^{n_0n_5,n_0n_5}_{(t_2-t_3)} \quad . \tag{4.9}$$

Using the same approximation as for the previous diagram, we see that it may be approximated by

$$\delta\Gamma = -\alpha\frac{g^2}{m^3\theta^2}\int_t\left(\int_x \varphi^*_{(x,t)}\star\varphi_{(x,t)}\right)^2 \quad , \quad \alpha > 0\ . \tag{4.10}$$

Another nonequivalent diagram of this class is represented in figure 8. Under the same approximation we used before, it contributes with:

$$\delta\Gamma = -\alpha\frac{g^2}{m^3\theta^2}\int_t\left(\int_x \varphi^*_{(x,t)}\star\varphi_{(x,t)}\right)^2 \quad , \quad \alpha > 0\ . \tag{4.11}$$

Thus under this approximation all non-planar contributions have the same expression. Numerical factors (we call α in equations (4.8), (4.10) and (4.11)) can of course be different.

There is also a planar diagram with one of its indexes not fixed by the external ones, figure 9. Because of this its contribution is more important than the previous ones:

$$\delta\Gamma = \frac{-S(2\pi\theta g)^2}{2(4!)^2}\delta_{(t_1-t_2)}\delta_{(t_3-t_4)}\delta^{n_0}_{n_2}\delta^{n_6}_{n_4}\delta^{n_3}_{n_5}\delta^{n_1}_{n_7}\sum_{\lambda\geq 0}\Delta^{\lambda n_3,\lambda n_3}_{(t_1-t_3)}\Delta^{\lambda n_1,\lambda n_1}_{(t_1-t_3)} \quad , \tag{4.12}$$

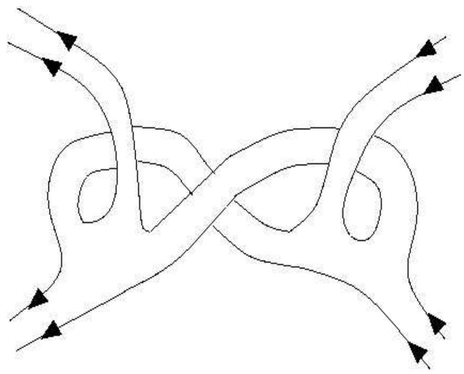

Figure 8: Non planar contribution to the four point function.

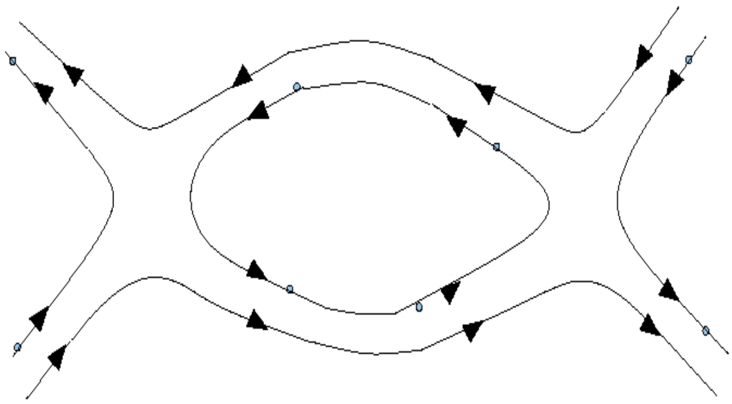

Figure 9: Planar contribution to the four point function.

which we again approximate, with the result:

$$\delta\Gamma = -\alpha(g\theta^{\frac{1}{2}})\mathcal{Z}_{(\frac{3}{2},\frac{2+\theta m^2}{2})}g\int\varphi^*\star\varphi\star\varphi^*\star\varphi\ ,\quad \alpha>0\ . \tag{4.13}$$

This diagram has a finite $\theta m^2 \to \infty$ limit. Indeed,

$$\lim_{x\to\infty}\sqrt{x}\mathcal{Z}_{(\frac{3}{2},\frac{2+x}{2})} = \beta\ , \tag{4.14}$$

where β is a positive number of order unity. So we have

$$\delta\Gamma = -\alpha\beta\frac{g^2}{m}\int\varphi^*\star\varphi\star\varphi^*\star\varphi\ ,\quad \alpha>0\,. \tag{4.15}$$

In this way, we see that only the last graph is leading when $\theta m^2 \to \infty$. This is a consequence of the free internal line (loop) which gives the most important contribution to the generating functional in this limit.

4.3 Approximate generating functional

Joining all the previous pieces, we get an approximate expression for the 1PI functional, in the $\theta m^2 \to \infty$ limit.

$$\Gamma = \mathcal{S} + \frac{g}{4!2\pi\sqrt{2\theta}} \int_{x,t} \varphi^* \star V\left(\frac{x}{\sqrt{\theta}}\right) \star \varphi + \varphi \star V\left(\frac{x}{\sqrt{\theta}}\right) \star \varphi^* - \\ -\alpha\beta\frac{g^2}{m}\int \varphi^* \star \varphi \star \varphi^* \star \varphi \ . \tag{4.16}$$

The approximation have been used to eliminate some of the four point contributions. Taking into account the form of the coefficients in the two point function it is easily verified that if the series which defines the one-body potential is truncated, this correction vanishes as well. We do not have, however, a closed analytical expression for that correction, so this term should be kept.

If further corrections are taken into account under the approximation $\theta m^2 \to \infty$ non-planar diagrams can be eliminated as in the four point function case. It is easily seen that if a series of internal lines connected to external legs are replaced by an internal loop the amplitude results a factor θm^2 bigger than the non-planar case. So, for example, if the two-loop self energy diagram is considered as in figure 4, the non-planar case would be suppressed by a factor $\mathcal{O}(\frac{1}{(\theta m^2)^2})$, and so the latter correction would not be important.

5. Non trivial vacuum configurations

Using the properties of the matrix base, exact classical solutions to the equations of motion can be found. A natural question is whether we can define a sensible quantum theory around those non trivial vacuum configurations. As we shall see, this is indeed the case. We will also analyze how the vacuum energy is shifted under variations of the parameters that characterize the solutions.

5.1 Classical solutions

Considering the real-time action associated to the Euclidean one of (2.5), we see that a classical solution must satisfy:

$$\ddot{\varphi} + m^2\varphi + \frac{1}{\theta^2}(x_\mu \star x_\mu \star \varphi) + \frac{1}{\theta^2}(\varphi \star x_\mu \star x_\mu) + \frac{2g}{4!}\varphi \star \varphi^* \star \varphi = 0 \ . \tag{5.1}$$

Using the ansatz

$$\varphi_{nk}(x,t) = e^{i\Omega_{nk}t} f_{nk}(x) \ , \tag{5.2}$$

we have a solution to the nonlinear problem if the following dispersion relation is satisfied:

$$\Omega_{nk}^2 = m^2 + \frac{2}{\theta}(n+k+1) + \frac{2g}{4!} \ . \tag{5.3}$$

This means that, at the classical level, objects with typical size θ can be stable (note the difference with the commutative case). There is a vast literature on the subject of solitonic solutions for noncommutative theories, some basic references are [1] and [16].

In order to study the quantum corrections, we deal next with the Euclidean version of the problem.

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5.2 Quantum case

Consider again the Euclidean action (2.5). The condition for an extremum with an ansatz such as (5.2) is

$$\Omega_{nk}^2 + m^2 + \frac{2}{\theta}(n+k+1) + \frac{2g}{4!} = 0 \,. \tag{5.4}$$

If we focus on time-independent solutions, a symmetry-breaking like potential is needed in order to have an extremum. We will, however, continue the discussion for a different kind of solution. As it may be easily verified, $\varphi = \eta f_{00}$ is a solution of the equation of motion if:

$$m^2 + \frac{2}{\theta} + \frac{2g\eta^2}{4!} = 0 \quad . \tag{5.5}$$

In the same way it is possible to generate more solutions of the form $\varphi = \eta f_{nk}$, with a non-linear condition for the amplitude. We will focus on the fundamental one ($\varphi = \eta f_{00}$) for an explicit analysis.

A first question is whether a generating functional (in the path integral formalism) can be constructed by expanding around this extremum. Next we want to know the dependence of the vacuum energy with the parameters of the problem. Let us first deal with the first (stability) condition. The second-order correction about the extremum of the Euclidean action is parameterized as follows:

$$\frac{1}{2}\begin{pmatrix}\chi \\ \chi^*\end{pmatrix}^{\dagger} \mathbb{H}(\mathcal{S}) \begin{pmatrix}\chi \\ \chi^*\end{pmatrix}, \tag{5.6}$$

where χ is the fluctuation around the non trivial solution, and $\mathbb{H}(\mathcal{S})$ is the Hessian matrix:

$$\begin{pmatrix} \frac{\delta^2\mathcal{S}}{\delta\varphi_1\delta\varphi_2^*} & \frac{\delta^2\mathcal{S}}{\delta\varphi_1^*\delta\varphi_2^*} \\ \frac{\delta^2\mathcal{S}}{\delta\varphi_1\delta\varphi_2} & \frac{\delta^2\mathcal{S}}{\delta\varphi_1^*\delta\varphi_2} \end{pmatrix}, \tag{5.7}$$

with the usual notation for kernels. So the consistency condition is equivalent to check that all eigenvalues of this matrix are positive. In fact we already have a basis of eigenvectors $\{\varphi/\varphi = e^{i\Omega t} f_{nk}(x),\ n,k \in \mathbb{N},\ \Omega \in \mathbb{R}\}$, and the eigenvalues are:

$$\begin{cases} \Omega^2 + m^2 + \frac{2}{\theta}(n_1+n_2+1) & n_1, n_2 \geq 1 \\ \Omega^2 + m^2 + \frac{2}{\theta}(n_1+n_2+1) + \frac{2g\eta^2}{4!} & n_{1,2} = 0, n_{2,1} \geq 1 \\ \Omega^2 + m^2 + \frac{2}{\theta} + \frac{6g\eta^2}{4!} & n_1 = n_2 = 0 \end{cases}$$

Using the condition (5.5) the set of eigenvalues is:

$$\begin{cases} \Omega^2 + \frac{2}{\theta}(n_1+n_2) - \frac{2g\eta^2}{4!} & n_1, n_2 \geq 1 \\ \Omega^2 + \frac{2}{\theta}(n_1+n_2) & n_{1,2} = 0, n_{2,1} \geq 1 \\ \Omega^2 + \frac{g\eta^2}{3!} & n_1 = n_2 = 0, \end{cases}$$

which are all positive if $g\eta^2 < \frac{2\,4!}{\theta}$.

Now the vacuum energy shift between two sets of parameters associated with the eigenvalues $\{\lambda'_n(\Omega)\}$ and $\{\lambda_n(\Omega)\}$ can be evaluated as:

$$\Delta E = \frac{1}{2\pi}\int_{d\Omega}\sum_n \log\left(\frac{\lambda'_n(\Omega)}{\lambda_n(\Omega)}\right) \quad . \tag{5.8}$$

This shows there is a way of changing the parameters such that the energy remains constant, if we mantain m and θ constant and if the product $g\eta^2$ does not change then $\Delta E = 0$. But note that we can change the coupling constant g and the amplitude of the solution η, with just one constraint.

In reference [17], a throughout study of non-trivial vacuum configurations in (real and complex) scalar models in 2 and 4 spacetime dimensions is presented. The kind of ansatz that we consider here may be regarded as an embedding to $2+1$ dimensions, of one of the solutions considered there for the 2-dimensional case.

6. Conclusions

We have shown explicitly that the self-dual model is a super-renormalizable theory, carrying out the explicit one-loop renormalization procedure, and evaluating the corresponding contributions to the effective action to that order. We have also found an approximate expression for the generating functional of proper vertices, under the assumption: $m^2\theta >> 1$.

Besides, some non trivial solutions in the presence of the GW term and a symmetry breaking potential have been found at classical level, and it was shown that they are stable under the leading quantum corrections, by evaluating the exact eigenvalues of the Hessian around those extrema. The resulting dependence of the vacuum energy on the model's parameters has also been explicitly found.

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A. The matrix basis

In this section we will briefly derive the properties of the basis which 'diagonalizes' Moyal product:

$$(f \star g)_{(x)} = f_{(x)} e^{\frac{i}{2}\theta_{\mu\nu}\overleftarrow{\partial_\mu}\overrightarrow{\partial_\nu}} g_{(x)} \quad . \tag{A.1}$$

First we build an operatorial representation of the algebra. Consider two hermitian operators such that $[x_1, x_2] = i\theta$, and define the creation an annihilation operators a and

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$a^\dagger$:

$$a = \frac{x_1 + ix_2}{\sqrt{2\theta}} \qquad [a, a^\dagger] = 1 \ . \tag{A.2}$$

To connect the algebra of functions $\mathcal{G}$ with the algebra of operators $\mathcal{G}'$ consider a map $\mathcal{S}^{-1} : f \in \mathcal{G} \to \mathcal{O}_{(f)} \in \mathcal{G}'$ defined by the following equation (here time is just a parameter):

$$\mathcal{O}_f(t) = \int_{\bar{k}\in\mathbb{R}^2} \frac{1}{(2\pi)^2} \hat{f}(\bar{k}, t) : e^{i\sqrt{\frac{\theta}{2}}(k^* a + k a^\dagger)} : \quad ; k = k_1 + ik_2 \ , \tag{A.3}$$

where :: denotes normal ordering and $\hat{f}$ is the usual Fourier transform:

$$\hat{f}(\bar{k}, t) = \int_{\bar{x}\in\mathbb{R}^2} f(\bar{x}, t) e^{-i\bar{k}_j \bar{x}_j} \ . \tag{A.4}$$

It is the work of a moment to verify the properties:

- $$\mathcal{O}_{f\star g} = \mathcal{O}_f \mathcal{O}_g \tag{A.5}$$

- $$Tr(\mathcal{O}_f) = \frac{1}{2\pi\theta} \int_x f(x, t) \tag{A.6}$$

- $$\mathcal{O}_{(\partial_{x_1} f)} = \left[\frac{i}{\theta} x_2, \mathcal{O}_f\right] \quad \mathcal{O}_{(\partial_{x_2} f)} = \left[-\frac{i}{\theta} x_1, \mathcal{O}_f\right] \ . \tag{A.7}$$

So the Moyal product in the algebra of functions is mapped to composition of operators. On the other hand, there is a special class of operators that allow a very easy way to perform composition, namely the ones which have the form $|n\rangle\langle k|$. So if we know $f_{nk} \in \mathcal{G}/\mathcal{O}_{(f_{nk})} = |n\rangle\langle k|$ we would have

$$f_{nk} \star f_{k'n'} = \delta_{kk'} f_{nn'} \quad f^*_{nk} = f_{kn} \ , \tag{A.8}$$

because of equation (A.5). This is the basis we mentioned above. To get an explicit form of the condition $\mathcal{O}_{f_{nk}} = |n\rangle\langle k|$ is enough to take matrix elements in equation (A.3) and use that Laguerre associated polynomials ($\mathbb{L}_j^{(n-j)}$) are complete (very useful identities can be found in [18]). Looking at the coefficients we find that the Fourier transform of such a function in polar coordinates is:

$$\hat{f}_{nj(\rho,\varphi)} = 2\pi\theta \sqrt{\frac{j!}{n!}} \left(i\sqrt{\frac{\theta}{2}}\right)^{j-n} e^{-i\varphi(n-j)} \rho^{j-n} e^{-\frac{\theta\rho^2}{4}} \mathbb{L}_j^{(n-j)}\left(\frac{\theta\rho^2}{2}\right) \ , \tag{A.9}$$

so for example a diagonal one is a Gaussian times a polynomial

$$\hat{f}_{nn(k)} = 2\pi\theta \, e^{-\frac{\theta k^2}{4}} \, \mathbb{L}^{(n)}\left(\frac{\theta k^2}{2}\right) \ . \tag{A.10}$$

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